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Continuous Wavelet Transform Reconstruction Factors for Selected Wavelets

General Background

This report expands on certain aspects of the analytical strategy for the Continuous Wavelet Transform (CWT) provided in *A Practical Guide to Wavelet Analysis* by Christopher Torrence and Gilbert P. Compo (Bulletin of the American Meteorological Society, 20 October 1997). Although the authors use geophysical examples in their work and although the target audience of the Bulletin of the American Meteorological Society is the meteorological community, the article is of substantial value for CWT users, is quoted throughout the wavelet analysis community, and is the basis for many CWT software implementations.

In *A Practical Guide to Wavelet Analysis* (the Guide) a method for reconstructing an original time series from the CWT is presented. The reconstructed signal is given by:

(Equation 1) $$\mathrm{x_n} = \frac{\delta_{\mathrm{j}} \delta_{\mathrm{t}}^{1/2}}{C_\delta \varphi_0(0)} \sum_{j=0}^{J} \frac{\Re\{W_n(s_j)\}}{s_j^{1/2}}$$

Where:

- δ_{j} is the spacing between scales
- δ_{t} is the time increment
- C_δ is the reconstruction factor
- $\varphi_0(0)$ is the normalized wavelet basis function evaluated at time zero
- J is the number of scales minus one
- $\Re\{W_n(s_j)\}$ is the real part of the wavelet transform
- s_j is the scale parameter ($s_0 2^{j\delta_j}$ where s_0 is the smallest scale used)

In the Guide the authors focus on the following three wavelet basis functions: Morlet wavelet, Paul wavelet, Derivative of Gaussian (DOG) wavelet. In their analysis the authors present a convention for normalizing the wavelet functions. The analytical forms for these normalized wavelet basis functions are reproduced below as they appear in the Guide:

The Morlet wavelet

$$\pi^{-1/4} e^{i\omega_0 \eta} e^{-\eta^2/2}$$

Where: η is the time parameter, ω_0 is the central frequency parameter

The Paul wavelet

$$\frac{2^m i^m m!}{\sqrt{\pi(2m)!}}(1 - i\eta)^{-(m+1)}$$

Where: η is the time parameter, m is the order parameter

The DOG wavelet

$$\frac{(-1)^{m+1}}{\sqrt{\Gamma(m + \frac{1}{2})}}\frac{d^m}{d\eta^m}(e^{-\eta^2/2})$$

Where: η is the time parameter, m is the derivative and

$\frac{d^m}{d\eta^m}(e^{-\eta^2/2})$ is evaluated as a probabilists' Hermite polynomial

In the Guide the authors utilize the Morlet wavelet with $\omega_0 = 6$, the Paul wavelet with order = 4 and the DOG with derivatives of 2 and 6. They provide reconstruction factors for these wavelets as well as $\varphi_0(0)$ values. The Guide further describes the *Fourier factor* to be used in converting scale values to the more useful frequency values.

This paper presents a table of reconstruction factors, $\varphi_0(0)$ values, and Fourier factor values for each of these wavelet functions using selected other values for the parameters of central frequency, order, and derivative. A test signal is evaluated using the parameters obtained and an error is provided. It also presents the methods for obtaining these values.

Summary of Results

Table I, Morlet wavelet

Parameter ω_0	$\varphi_0(0)$	C_δ	Fourier Factor	Test signal error*
5	0.7511	0.9484	1.2325	0.0070
5.336**	0.7511	0.8831	1.1574	0.0070
6	0.7511	0.7784	1.0330	0.0070
7	0.7511	0.6616	0.8886	0.0070
8	0.7511	0.5758	0.7794	0.0070
10	0.7511	0.4579	0.6252	0.0070
12	0.7511	0.3804	0.5218	0.0070
14	0.7511	0.3254	0.4477	0.0070
16	0.7511	0.2844	0.3919	0.0070
20	0.7511	0.2272	0.3138	0.0070

*relative residual of reconstructed test signal, **$\pi\sqrt{2/\ln(2)}$

Table II, Paul wavelet

Order m	$\varphi_0(0)$	C_δ	Fourier Factor	Test signal error
4	1.0789	1.1330	1.3963	0.0070
5	1.1373i	-0.9065i	1.1424	0.0070
6	-1.1879	-0.7554	0.9666	0.0070
7	-1.2327i	0.6475i	0.7392	0.0070
8	1.2731	0.5665	0.5984	0.0070
10	-1.3441	-0.4532	0.4054	0.0070
16	1.5081	0.2833	0.3065	0.0070
20	1.5934	0.2266	0.2464	0.0070
30	-1.7616	-0.1511	0.2060	0.0070
40	1.8919	0.1133	0.1551	0.0070

Table III, DOG wavelet

Derivative m	$\varphi_0(0)$	C_δ	Fourier factor	Test signal error
2	0.8673	3.5987	3.9738	0.0075
4	-0.8796	2.4014	2.9619	0.0068
6	0.8841	1.9212	2.4645	0.0065
8	-0.8863	1.6467	2.1551	0.0065
12	-0.8886	1.3307	1.7772	0.0065
16	-0.8898	1.1464	1.5468	0.0065
20	-0.8905	1.0222	1.3877	0.0065
30	0.8914	0.8312	1.1377	0.0065
40	-0.8918	0.7183	0.9873	0.0065
60	-0.8923	0.5853	0.8078	0.0065

Determinative Methods

The values for $\varphi_0(0)$ were obtained by evaluating the wave functions at $\eta = 0$. Fourier factors were obtained using the formulas provided in the Guide. The values for C_δ were obtained by performing a CWT on a delta function at $\eta = 0$ using the appropriate wavelet parameters. The quantity:

$$\sum_{j=0}^{J} \frac{\Re\{W_n(s_j)\}}{s_j^{1/2}}$$

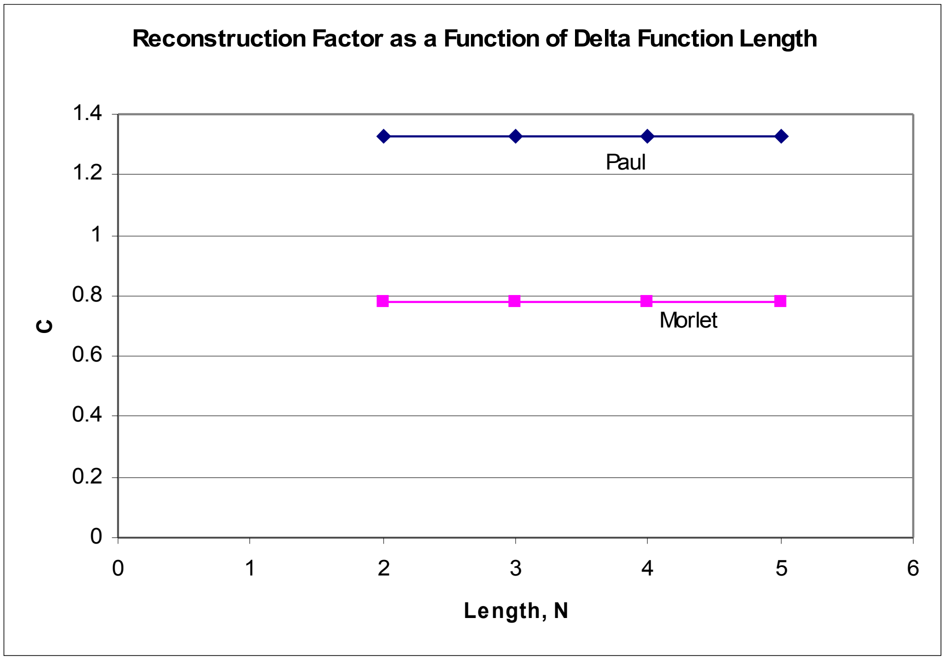
was evaluated for each time in the calibration signal. Subsequently, the following rearranged reconstruction factor equation was evaluated, bearing in mind that the maximum value for the reconstructed waveform must be unity.

(Equation 2) $C_\delta = \frac{\delta_j \delta_t^{1/2}}{\varphi_0(0)} \left(\sum_{j=0}^{J} \frac{\Re\{W_n(s_j)\}}{s_j^{1/2}} \right)_{MaxAbs}$ where

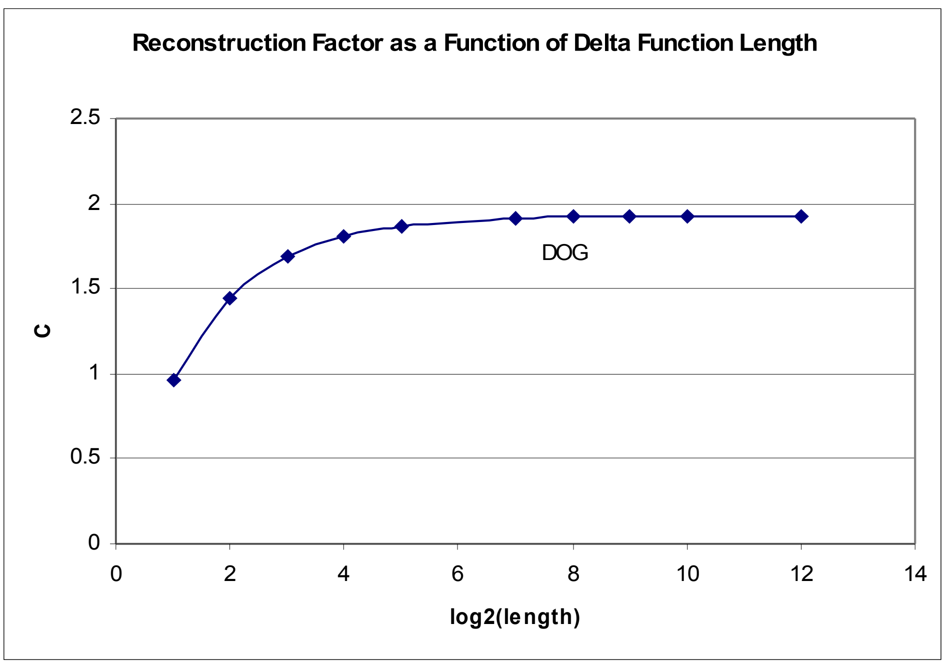
$\left(\sum_{j=0}^{J} \frac{\Re\{W_n(s_j)\}}{s_j^{1/2}} \right)_{MaxAbs}$ is the value of the sum that has maximal absolute value

Delta function length

In the case of the Morlet and Paul wavelets a delta function of length N = 2 was adequate for obtaining a value for C_δ . The DOG wavelet required a function of length N = 256. The following graph illustrates that the reconstruction factor for the Paul and Morlet inverse CWT has reached a limiting value with a two point delta function.

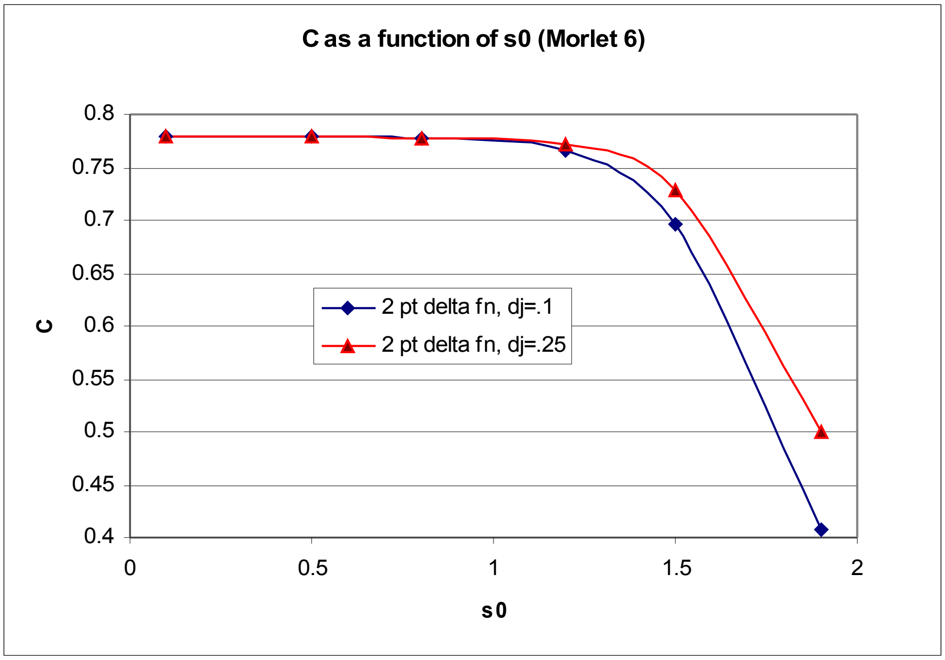


The graph below illustrates that the reconstruction factor for the DOG inverse CWT has reached a limiting value with a 256 point delta function.



Choice of s_0 and δ_j

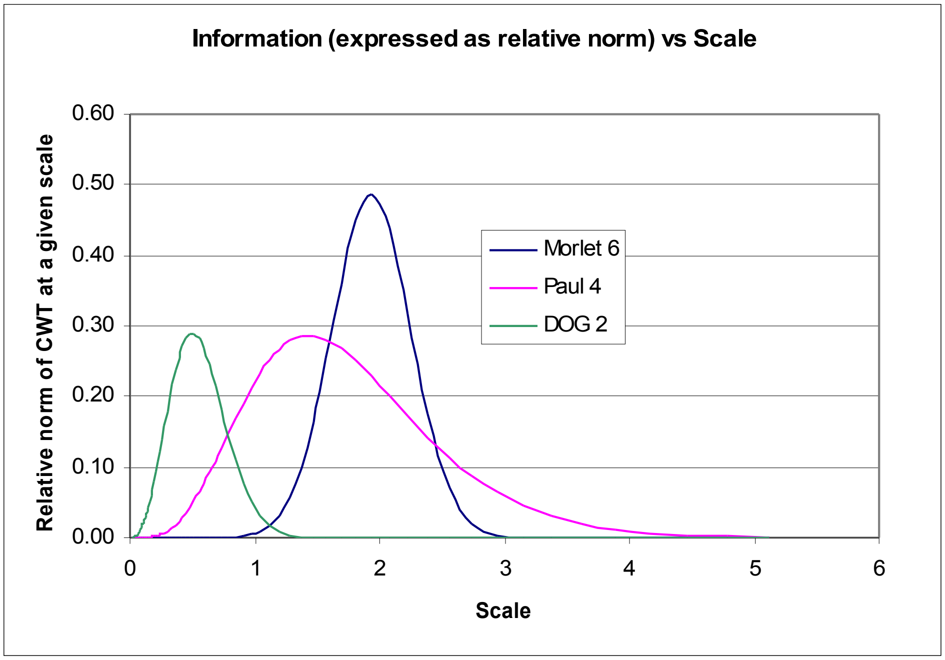
Values of s_0 below the value corresponding to an equivalent Fourier period of $2\delta_t$ have limited meaning in the CWT domain. However, in performing a reconstruction, a more exact representation can be obtained from a CWT that extends lower in scale to capture more information. This is also true of the calibration function used to obtain the reconstruction factor. In evaluating reconstruction factors it is necessary to verify that the choices for s_0 and δ_j enable the level of information-capture necessary to obtain a reasonable approximation to the limiting value for C_δ (see graph below).



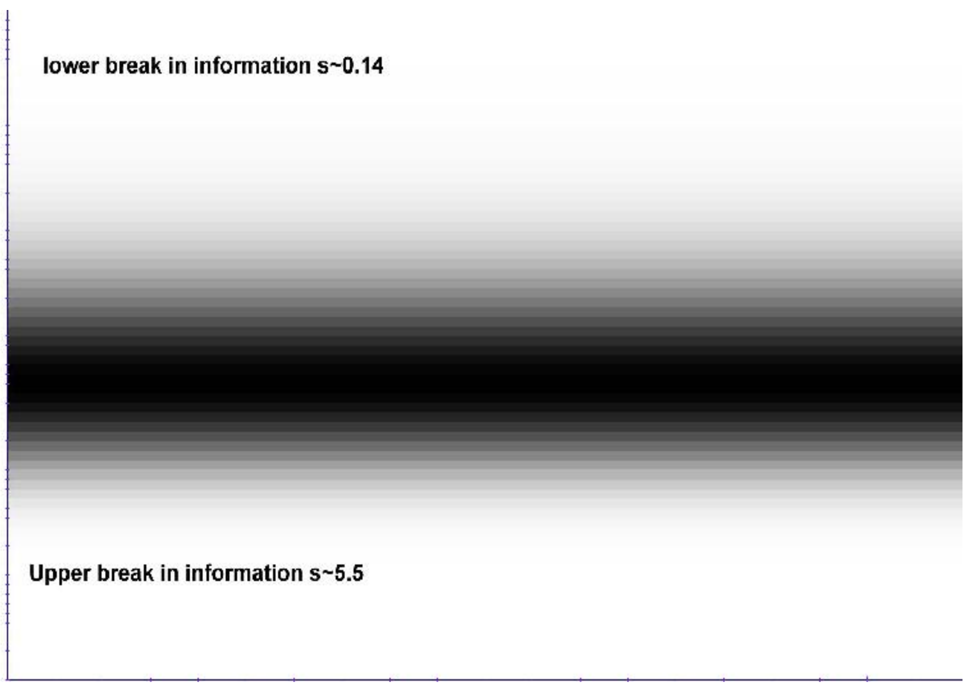
Another approach to evaluating the level of information present at a given scale is to evaluate the relative norm of each scale vector (of length N) in the CWT. In the following graph the relative norm for each scale of the CWT for a delta function of length N=2 is plotted vs. scale where:

(Equation 3) $$\text{relative norm at scale s} = \frac{\left\| \Re\{CWT(s)\} \right\|_2}{\left\| \Re\{CWT\} \right\|_F}$$

where F indicates the Frobenius norm



Finally, another way to evaluate effective choice of s_0 and the total number of scales J is to inspect the CWT's scalogram. Ordinarily we would plot $\Re\{CWT\}^2$ but here we plot the absolute value of the real CWT to improve contrast for low level information. Also, to enhance contrast, plotting in black and white is the preferred method.



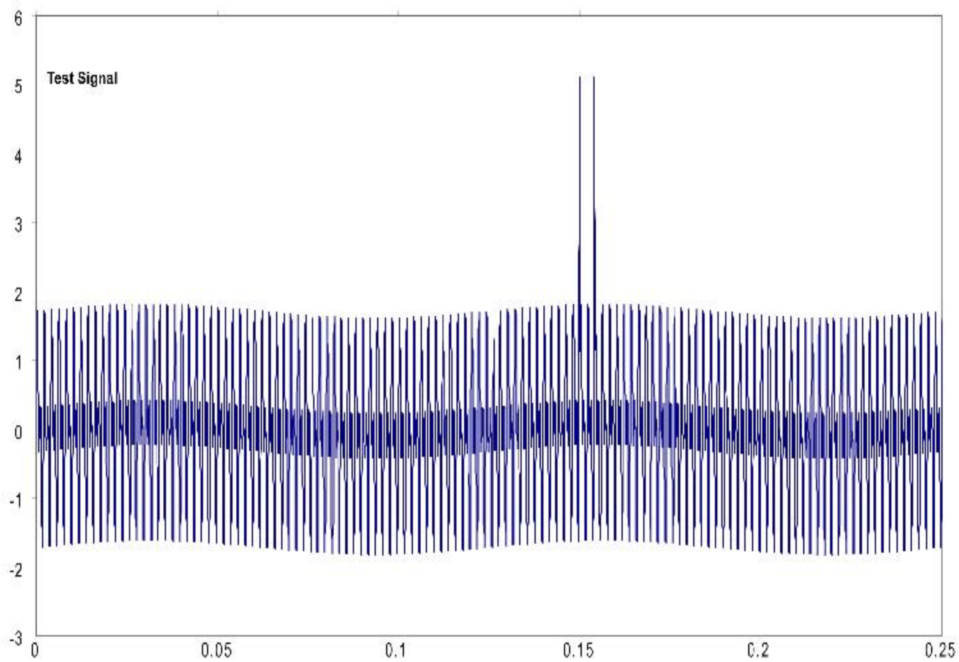
Performance Evaluation

To evaluate the reconstruction factors a test signal was prepared and transformed using the various wavelet/parameter combinations. The signals were then reconstructed using Equation 1 together with the values for C_δ and $\varphi_0(0)$ provided in the Summary of Results, Tables I, II, and III.

The test signal consisted of:

$\delta_t = 0.000125$, N=2048

- $\sin(2\pi\, 500t)$
- $\sin(2\pi\, 1000t)$
- $0.1\sin(2\pi\, 8t)$
- an impulse of amplitude 5 at t = 0.150
- an impulse of amplitude 5 at t = 0.154



CWT scalograms and scale-norm profiles were used to evaluate scale information levels, and s_0 and J were set to ensure optimal inclusion of CWT data. The resolution, δ_j, was set to 0.05. A relative residual error value was calculated from the reconstructed signal as follows:

(Equation 4) $$\text{relative residual} = \frac{\left\| x_t - \widehat{x_t} \right\|_2}{\left\| x_t \right\|_2}$$

where x_t is the original signal vector and $\widehat{x_t}$ is the reconstructed signal vector.

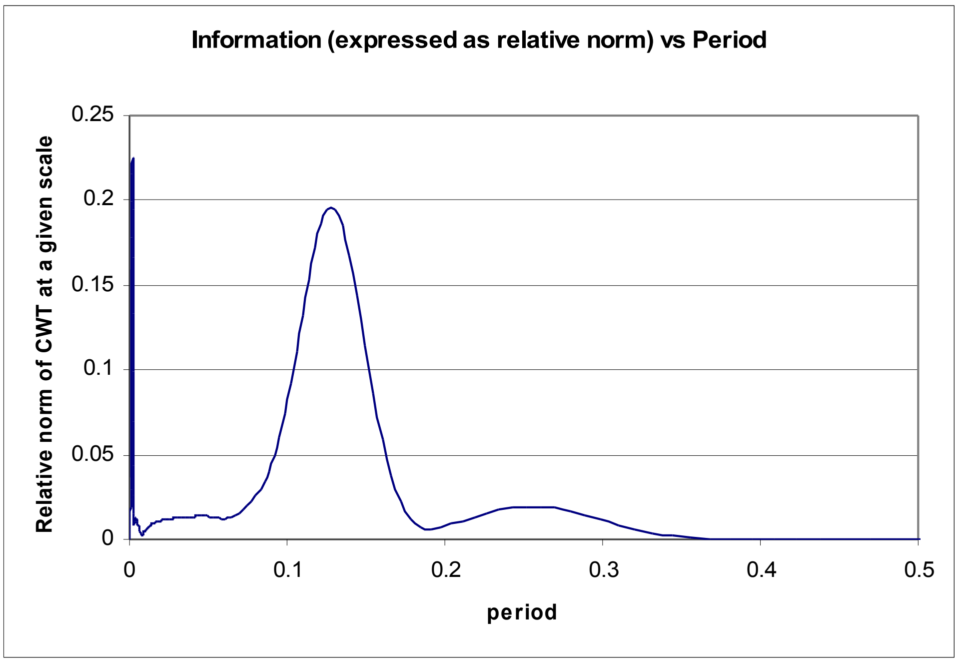
The results are included in the tables provided in the Summary of Results section. All values were approximately 0.007 or 0.7% relative error.

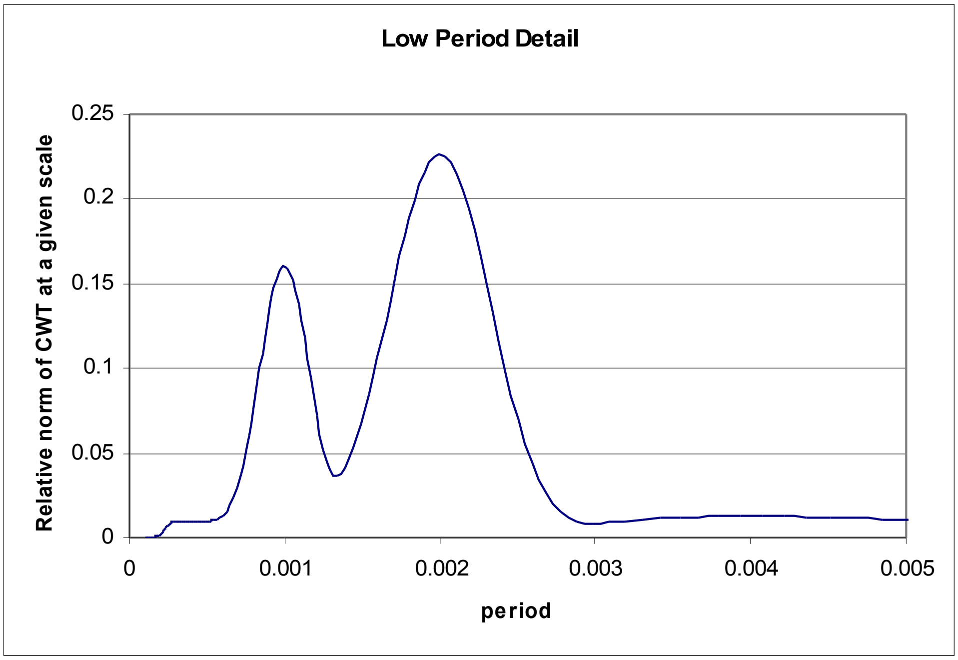
An Example Case

To illustrate certain aspects for obtaining a reconstruction factor using these methods, an example for the test signal evaluation for the Morlet wavelet with central frequency 6 is discussed.

The effect of s_0

The following are graphs of the relative norm of each scale vector in the CWT.

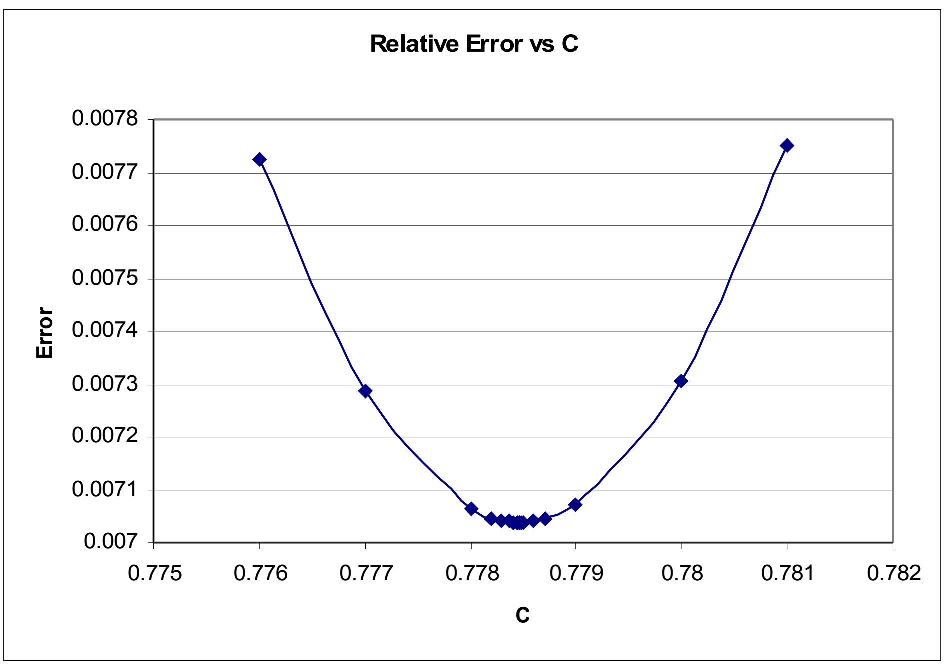




When the CWT is performed with s_0 set to a value corresponding to a period of $2\delta_t$, the residual based error value increases from 0.007 to 0.030 or 3% relative error.

Graphic determination of C_δ

If we plot the relative error of the reconstructed signal vs. C_δ we obtain a profile from which a minimum error producing reconstruction factor can be found.



This approach gives an optimal C_δ value of 0.7785. This compares well with the value of 0.7784 obtained using the methods described here.

Finally, the reconstruction factor for the Morlet 6 wavelet provided in the Guide is 0.776. This three place value is slightly lower than that obtained here. The value for C_δ of 0.776 yields a reconstructed signal with only slightly higher relative error (0.0077). Three place reconstruction factors, with uncertainty in the least significant digit, should be adequate for most applications.